Defect Stream Function, Law-of-the-Wall/Wake Method for Turbulent Boundary Layers

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The application of the defect stream function to the solution of the two-dimensional, incompressible boundary layer problem is reexamined. A law-of-the-wall/law-of-the-wake formulation for the inner part of the boundary layer is presented that greatly simplifies the computational task near the wall and eliminates the need for an eddy viscosity model in this region. The eddy viscosity model in the outer part of the boundary layer is arbitrary. Formulations for both equilibrium and nonequilibrium boundary layers are presented, and results are compared with previous methods for equilibrium boundary layers. The present treatment eliminates the need for resolving the flow in the inner part of the boundary layer computationally, thereby improving computational efficiency.

Introduction

THE law of the wall for turbulent boundary layers, with the law of the wake as a corollary, is one of the most well established, widely accepted, and accurate empiricisms in fluid mechanics. The general purpose of the present research is to exploit the law of the wall and law of the wake in order to improve both the accuracy and the efficiency of computational fluid dynamics. These laws are used to model the flow in the inner part of turbulent boundary layers where they are most applicable; the outer or defect part of the boundary layer is solved computationally. The increase in efficiency results from the elimination of the need to resolve the inner, or high gradient, part of the boundary layer computationally; and the increase in accuracy is due to the replacement of inner layer eddy viscosity empiricisms with more accurate law-of-the-wall/law-of-the-wake empiricisms.

In the initial effort at solving this problem,[1] the present authors combined an integral treatment of the inner layer with a finite-difference treatment of the outer layer. The critical implementation feature was the algorithm developed to interact the inner and outer treatments. The problem was formulated in terms of the conventional stream function, and solutions were obtained for incompressible flow over a flat plate, a two-dimensional circular cylinder, and two-dimensional elliptic cylinders. Most of the results of Ref. 1 are for the lowest order approximation for pressure-gradient effects, which ignores these effects in the inner layer. These results clearly indicate the need to account for pressure gradient effects in the inner region when these gradients are large, such as in the case of circular and elliptic cylinders. Although successful computations were made and reported in Ref. 1 in which pressure gradient effects were accounted for in the inner region, a fair amount of numerical difficulty was encountered in these computations. In the present effort, it was found that numerical problems of the type encountered can be reduced considerably if computational steps in the interaction algorithm are replaced, when possible, with equivalent analytical steps.

The present analysis is formulated in terms of the defect stream function used by Mellor and Gibson[2] and validated experimentally by Clauser[3] rather than the conventional stream function. The defect stream function has been chosen because it properly characterizes the flow in the outer region of the boundary layer, which must be solved numerically, and because it has the remarkable property of having a first integral, to lowest order in the nondimensional shear stress velocity ratio, in the direction normal to the surface. This property facilitates the implementation of the integral treatment in the inner region.

Two major problems with the defect stream function formulation encountered in Ref. 2 concerned the enforcement of law of the wall behavior near the wall and the implementation of the wall-layer eddy viscosity model. The computational complexity Mellor and Gibson encountered in the inner region probably explains the general lack of popularity of the defect stream function formulation. The present approach completely overcomes this complexity.

There are several other current methods that use a law-of-the-wall formulation in the inner region. Wall function methods such as that of Viegas et al.[4] Simply patch the numerical and analytic solutions at the first grid point and employ an eddy viscosity model throughout the boundary layer. The recent method of Walker et al.[5] is similar to the method of Ref. 1 in that it uses the conventional stream function and interacts inner layer analytic solutions with outer layer numerical solutions at a point determined in the analysis. Wilcox[6] employs the defect stream function formulation but uses an eddy viscosity across the entire boundary layer; he does not make use of the first integral property of the defect stream function formulation. None of these methods includes the wake function.

Defect Stream Function Formulation

The basic formulation parallels that of Mellor and Gibson.[2] However, the present development includes nonequilibrium boundary layers; and the eddy viscosity model for the present treatment is much more general than that of Ref. 2. In fact, it is unnecessary to specify the eddy viscosity in the inner layer at all.

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Basic Equations

The governing equations for incompressible turbulent boundary layer flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{\partial}{\partial y}\left[(v + v_t)\frac{\partial u}{\partial y}\right] \tag{2}$$

where x and y are the normal and tangential coordinates, u and v the respective velocity components, U the edge velocity, and ν the kinematic viscosity. The sum of ν and the eddy viscosity is

$$\nu + \nu_t = K(x,y)\delta^* U \tag{3}$$

where K is a general nondimensional function of x and y and δ^* is the displacement thickness.

In this treatment the defect stream function of Clauser[3] is used. This function is defined as

$$f'(\xi,\eta) = \frac{u - U}{u^*} \tag{4}$$

where

$$\xi = x, \qquad \eta = \frac{y}{\Delta}$$

The prime denotes differentiation with respect to η, and the shear stress velocity u^* is defined as

$$u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

where ρ and τ_w are the density and the shear stress at the wall. The boundary layer thickness parameter Δ is defined as

$$\Delta = -\int_0^\infty f'(\xi,\eta)\,\mathrm{d}y = \frac{U}{u^*}\int_0^\infty \left(1 - \frac{u}{U}\right)\mathrm{d}y = \frac{U\delta^*}{u^*}$$

or

$$u^*\Delta = U\delta^* \tag{5}$$

Partial derivatives with respect to x and y are of the form

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \eta\frac{\dot{\Delta}}{\Delta}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial y} = \frac{1}{\Delta}\frac{\partial}{\partial \eta} \tag{6}$$

where the dot represents differentiation with respect to ξ.

Law of the Wall and Wake

It is assumed that a law of the wake and wall for the inner part of the boundary layer is known. This law is of the form

$$\frac{u}{u^*} = g(y^+) + h(\xi,\eta) \tag{7}$$

where $g(y^+)$ is the wall function and $h(\xi,\eta)$ is the wake function. The inner variable y^+ and the Clauser pressure gradient parameter β are defined as

$$y^+ = \frac{u^* y}{\nu} = Re_{\delta^*}\eta \tag{8a}$$

$$\beta = \frac{\delta^*}{\tau_w}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{8b}$$

where p is the pressure and the Reynolds number based on the edge velocity, and the displacement thickness is

$$Re_{\delta^*} = \frac{U\delta^*}{\nu} = \frac{u^*\Delta}{\nu}$$

For nonequilibrium flow, the parameter β is a function of x and hence ξ; for equilibrium flow, the parameter β is a constant. In the present treatment it is assumed that the law of the wall is of the form

$$g(y^+) = \frac{1}{\kappa}\ln y^+ + B \tag{9}$$

Shear Stress Velocity Ratio

The nondimensional shear stress velocity is defined as

$$\gamma = \frac{u^*}{U}$$

In the present treatment, the ratio γ is evaluated with the law of the wall/wake. From Eqs. (4) and (7), it is seen that at the match point η_m between the inner and outer parts of the boundary layer u can be expressed as

$$u = u^*\left[g\left(\frac{U\gamma\Delta}{\nu}\eta_m\right) + h(\xi,\eta_m)\right] = U + u^* f'(\xi,\eta_m)$$

The equation for γ used in this paper is obtained from this equation as

$$\gamma = (g_m + h_m - f'_m)^{-1} \tag{10}$$

In the analysis that follows an expression for the gradient of γ with respect to ξ is needed. Differentiation of the expression above yields

$$\dot{\gamma} = -\gamma^2\left[\left(\frac{\mathrm{d}g}{\mathrm{d}y^+}\right)_m \dot{y}^+_m + \dot{h}_m - \dot{f}'_m\right] \tag{11}$$

With Eqs. (8) and (9), the expression

$$\frac{\mathrm{d}g}{\mathrm{d}y^+}\dot{y}^+ = \frac{\mathrm{d}g}{\mathrm{d}y^+}y^+\left(\frac{\dot{\gamma}}{\gamma} + \frac{\dot{U}}{U} + \frac{\dot{\Delta}}{\Delta}\right) = \frac{1}{\kappa}\left(\frac{\dot{\gamma}}{\gamma} + \frac{\dot{U}}{U} + \frac{\dot{\Delta}}{\Delta}\right)$$

is obtained. Equation (11) can be written as

$$\frac{\dot{\gamma}U}{\gamma\dot{U}} = -\frac{\gamma/\kappa}{1 + \gamma/\kappa}\left(1 + \frac{\dot{\Delta}U}{\Delta\dot{U}}\right) + \frac{\Delta}{\beta}\frac{\dot{h}_m - \dot{f}'_m}{1 + \gamma/\kappa} \tag{12}$$

The derivative $\dot{\eta}_m$ has been neglected because, as can be seen later, the lowest order term in η_m is small and generally does not vary with ξ.

Governing Equation

Now one equation for the defect stream function f can be written. With Eqs. (4) and (6), the gradients of u with respect to x and y are found to be

$$\frac{\partial u}{\partial x} = \dot{U}(1 + \gamma f') + U\dot{\gamma}f' + U\gamma\dot{f}' - \frac{\dot{\Delta}}{\Delta}\eta U\gamma f''$$

$$\frac{\partial u}{\partial y} = \frac{U\gamma}{\Delta}f''$$

From Eq. (1), the continuity equation, the v component of velocity is found to be

$$v = -\int_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y = -\dot{U}\Delta(\eta + \gamma f) - U\Delta\dot{\gamma}f + \dot{\Delta}U\gamma(\eta f' - f) - \Delta U\gamma\dot{f} \tag{13}$$

Equation (2), the tangential momentum equation, is

$$\frac{1}{\beta}(Kf'')' - \left(1 + \frac{\dot{\Delta} U}{\Delta \dot{U}}\right) f''(\eta + \gamma f) + \frac{\dot{\gamma} U}{\gamma \dot{U}} \{f' + \gamma[(f')^2 - ff'']\} + 2f' + \gamma (f')^2 = \frac{\Delta}{\gamma \beta} \dot{f}'(1 + \gamma f') - \frac{\Delta}{\beta} \dot{f} f''$$

Let s be a nondimensional tangential coordinate of the form

$$s = \int^{\xi} \frac{\gamma}{\Delta} \mathrm{d}\xi$$

With this definition, and Eq. (12), the tangential momentum equation can be written as

$$\frac{1}{\beta}(Kf'')' - \left(1 + \frac{\dot{\Delta} U}{\Delta \dot{U}}\right)\left(f''(\eta + \gamma f) + \frac{\gamma/\kappa}{1 + \gamma/\kappa} \times \{f' + \gamma[(f')^2 - ff'']\}\right) + \frac{\gamma/\beta}{1 + \gamma/\kappa}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right) \times \{f' + \gamma[(f')^2 - ff'']\} + 2f' + \gamma (f')^2 = \frac{1}{\beta}\left[(1 + \gamma f')\frac{\partial f'}{\partial s} - \gamma \frac{\partial f}{\partial s} f''\right] \tag{14}$$

Boundary Conditions

There are three boundary conditions for f. From Eq. (13) for v, the normal flow boundary condition at the surface, $v = 0$ at $\eta = 0$, is observed to correspond to

$$f = 0, \qquad \eta = 0$$

The value of f_∞ is obtained from the definition of the displacement surface:

$$U\delta^* = \int_0^\infty (U - u)\, \mathrm{d}y = -u^*\Delta \int_0^\infty f'\, \mathrm{d}\eta = u^*\Delta[f(0) - f_\infty]$$

It follows that

$$f_\infty = -1 \tag{15}$$

The final boundary condition involves the shear stress at the wall. The condition is

$$\frac{\tau_w}{\rho} = \lim_{y \to 0} (v + v_t)\frac{\partial u}{\partial y}$$

or

$$\lim_{\eta \to 0} K(\xi,\eta) f''(\xi,\eta) = 1 \tag{16}$$

These boundary conditions are the same as those of Mellor and Gibson[2] if $K(\xi,\eta)$ is replaced by the Clauser constant k.

First Integral of Governing Equation

Equation (14) can be written as

$$\left\{\frac{K}{\beta} f'' - \left(1 + \frac{\dot{\Delta} U}{\Delta \dot{U}}\right)\left[\eta f' - \frac{(1 - \gamma f')f}{1 + \gamma/\kappa}\right] + \frac{\gamma/\beta}{1 + \gamma/\kappa}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right)(1 - \gamma f')f + 2f - \frac{1}{\beta}\frac{\partial f}{\partial s} + \frac{\gamma}{\beta}\frac{\partial f}{\partial s} f'\right\}' = -\gamma\left[1 + \left(1 + \frac{\dot{\Delta} U}{\Delta \dot{U}}\right)\frac{1 - \gamma/\kappa}{1 + \gamma/\kappa} + 2\frac{\gamma/\beta}{1 + \gamma/\kappa}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right)\right](f')^2 + \frac{\gamma}{\beta}\frac{\partial}{\partial s}(f')^2$$

With the boundary condition (16), this equation can be integrated across the boundary layer to evaluate the quantity $1 + (\dot{\Delta} U/\Delta \dot{U})$. This value is

$$1 + \frac{\dot{\Delta} U}{\Delta \dot{U}} = -\frac{\left(1 + \frac{\gamma}{\kappa}\right)\left(\frac{1}{\beta} + 2 - \gamma G + \frac{\gamma}{\beta}\frac{\mathrm{d}G}{\mathrm{d}s}\right) + \frac{\gamma}{\beta}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right)(1 - 2\gamma G)}{1 - \gamma G\left(1 - \frac{\gamma}{\kappa}\right)} = 1 + \frac{1}{m(s)} \tag{17}$$

where

$$G = \int_0^\infty (f')^2\, \mathrm{d}\eta$$

Therefore, the integral of the governing equation for arbitrary η is

$$(1 - \gamma f')\frac{\partial f}{\partial s} = Kf'' - \beta\left(1 + \frac{1}{m}\right)\left[\eta f' - \frac{1 - \gamma f'}{1 + \gamma/\kappa} f\right] + 2\beta f - 1 + \gamma\left[\beta + \beta\left(1 + \frac{1}{m}\right)\frac{1 - \gamma/\kappa}{1 + \gamma/\kappa} + 2\frac{\gamma}{1 + \gamma/\kappa}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right)\right]\int_0^\eta (f')^2\, \mathrm{d}\eta - \gamma\frac{\partial}{\partial s}\int_0^\eta (f')^2\, \mathrm{d}\eta + \frac{\gamma}{1 + \gamma/\kappa}\left(\frac{\mathrm{d}h_m}{\mathrm{d}s} - \frac{\mathrm{d}f'_m}{\mathrm{d}s}\right) \times (1 - \gamma f')f \tag{18}$$

The equilibrium form of this equation is the same as the governing equation of Mellor and Gibson.[2]

As noted in Ref. 2, the nondimensional shear stress velocity ratio γ is a small parameter. As a result, the defect stream function can be expanded in terms of γ as

$$f = f_0 + \gamma f_1 + \cdots$$

The zero order forms of Eqs. (17) and (18) are

$$1 + \frac{1}{m} = -\left(\frac{1}{\beta} + 2\right)$$

$$\frac{\partial f_0}{\partial s} = Kf''_0 + (1 + 2\beta)\eta f'_0 - f_0 - 1 \tag{19}$$

Formulation for Large β

This treatment corresponds to that of Ref. 2. It covers the case of adverse pressure gradient. Mellor and Gibson define a "pressure velocity" as

$$u_p = \left(\frac{\delta^*}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}\right)^{1/2}$$

This quantity is related to u^* and β as

$$u_p = \beta^{1/2} u^*$$

Now define the quantities

$$S = \beta s, \qquad N = \beta^{1/2}\eta, \qquad F(S,N) = f(s,\eta), \qquad \lambda = \beta^{1/2}\gamma$$

It follows that

$$F'(S,N) = \frac{f'(s,\eta)}{\beta^{1/2}} = \frac{u - U}{u_p}, \qquad F''(S,N) = \frac{f''(s,\eta)}{\beta}$$

$$H(S,N) = \frac{h(s,\eta)}{\beta^{1/2}}$$

The first integral of the governing equation is

$$(1-\lambda F')\frac{\partial F}{\partial S}=KF''-\left(1+\frac{1}{m}\right)\left[NF'-\frac{(1-\lambda F')F}{1+\lambda/(\beta^{1/2}\kappa)}\right]$$
$$+\frac{\lambda}{1+\lambda/(\beta^{1/2}\kappa)}\left(\frac{\mathrm{d}H_m}{\mathrm{d}S}-\frac{\mathrm{d}F'_m}{\mathrm{d}S}\right)(1-\lambda F')F+2F-\frac{1}{\beta}$$
$$+\lambda\left[1+\left(1+\frac{1}{m}\right)\frac{1-\lambda/(\beta^{1/2}\kappa)}{1+\lambda/(\beta^{1/2}\kappa)}+\frac{2\lambda}{1+\lambda/(\beta^{1/2}\kappa)}\right.$$
$$\left.\times\left(\frac{\mathrm{d}H_m}{\mathrm{d}S}-\frac{\mathrm{d}F'_m}{\mathrm{d}S}\right)\right]\int_0^N(F')^2\,\mathrm{d}N-\lambda\frac{\partial}{\partial S}\int_0^N(F')^2\,\mathrm{d}N \tag{20}$$

The boundary conditions are

$$F(0)=0,\qquad F_\infty=-1$$

Here, and in Ref. 2, these forms are used for values of β greater than one.

Inner Region Treatment

It is this treatment that most distinguishes the present method of solution from that of Mellor and Gibson.[2] In effect, the treatment replaces the surface boundary condition with a condition at the boundary between the outer and inner regions of the boundary layer. The inner region is defined as that part of the boundary layer where the empirical law of the wall and wake pertain. The outer region is defined as that part of the boundary layer where the outer eddy viscosity model pertains. In general, there is one point, the "match point," where both the law of the wall and wake and the outer eddy viscosity model are both correct.

The term "match point" should not be construed to mean that the inner and outer solutions are being matched in the formal sense. Actually, these solutions are being patched at one point. As discussed in Ref. 1, this procedure is completely analogous to patching the inner and outer eddy viscosity models in zero-equation formulations.

It should be noted that the inclusion of the wake function in the inner layer empirical expression means that the match point is not confined to the logarithmic part of the boundary layer.

Equation Relating f and f'

An equation can be established that relates f and f' throughout the inner part of the boundary layer. This equation is obtained by integrating Eq. (4) by parts and using Eqs. (7) and (9). The result is

$$f=\eta\left(f'-\frac{1}{\kappa}\right)-\int_0^\eta \eta h'\,\mathrm{d}\eta \tag{21}$$

The use of this equation at the match point ensures the continuity of f and f'. The match point is positioned so that the derivative f'' is continuous.

Match Point Location

The match point location is determined with Eq. (18), the first integral of the tangential momentum equation, and Eqs. (7) and (9) for the law of the wall and wake, which are used to evaluate the f'' term in Eq. (18). The terms containing derivatives with respect to s, which are small in the inner part of the boundary layer where the match point is located, are simply neglected. Equation (21) is used to relate f to f' and the wake function. With some manipulation, the equation for the match point location η_m is found to be

$$\frac{K}{\kappa}=\eta_m-\frac{\beta}{1+\gamma/\kappa}\left\{2\left(1-\frac{\gamma}{m\kappa}\right)\left(f'_m-\frac{1}{\kappa}\right)-\frac{1+1/m}{\kappa}\right.$$
$$\left.+\gamma\left(1-\frac{\gamma}{m\kappa}\right)\left[\left(f'_m-\frac{1}{\kappa}\right)^2+\frac{1}{\kappa^2}\right]\right\}\eta_m^2-K\eta_m h'(\eta_m)$$
$$+\frac{\beta}{1+\gamma/\kappa}\left[\left(3+\frac{1}{m}\right)+\left(1-\frac{1}{m}\right)\frac{\gamma}{\kappa}-\gamma\left(1+\frac{1}{m}\right)\right.$$
$$\left.\times\left(f'_m-\frac{1}{\kappa}\right)\right]\eta_m\int_0^{\eta_m}\eta h'(\eta)\,\mathrm{d}\eta+2\frac{\gamma\beta}{1+\gamma/\kappa}\left[\left(1+\frac{1}{m}\right)\right.$$
$$\left.+\left(1-\frac{\gamma}{m\kappa}\right)\right]\eta_m\int_0^{\eta_m}\left(f'-\frac{1}{\kappa}\right)\eta h'(\eta)\,\mathrm{d}\eta \tag{22}$$

Now consider the wake function. One of the most widely accepted wake functions is that of Moses[7]:

$$h(\xi,\eta)=\frac{2}{\kappa}\,\Pi(\beta)\left[3\left(\frac{\Delta\eta}{\delta}\right)^2-2\left(\frac{\Delta\eta}{\delta}\right)^3\right] \tag{23}$$

where δ is the boundary-layer thickness. The ratio Δ/δ and the coefficient Π are functions of β, and β is a function of ξ. White[8] approximates Π as

$$\Pi=\frac{4}{5}\left(\beta+\frac{1}{2}\right)^{3/4} \tag{24}$$

Because the wake function in Eq. (23) is of order η^2 near the wall, all of the terms in Eq. (22) with integrals are of order $\gamma\eta_m^4$ near the wall except one, which is of order η_m^4. It will be shown that γ and η_m are small. Therefore, it is concluded that the integral terms in Eq. (22) can be neglected.

The equation for the match point location is

$$A\eta_m^2-\eta_m+\frac{K}{\kappa}=0 \tag{25}$$

where

$$A=\frac{\beta}{1+\gamma/\kappa}\left\{2\left(1-\frac{\gamma}{m\kappa}\right)\left(f'_m-\frac{1}{\kappa}\right)-\frac{1+1/m}{\kappa}\right.$$
$$\left.+\gamma\left(1-\frac{\gamma}{m\kappa}\right)\left[\left(f'_m-\frac{1}{\kappa}\right)^2+\frac{1}{\kappa^2}\right]\right\}+\frac{12}{\kappa}\frac{\Pi K}{(\delta/\Delta)^2}\left(1-\frac{\Delta\eta_m}{\delta}\right)$$

and where the wake function of Moses has been used. The lowest order approximation for A in the limit of small γ is

$$A=2\beta f'_m+\frac{1}{\kappa}+\frac{12}{\kappa}\frac{\Pi K}{(\delta/\Delta)^2}$$

The solution for η_m is

$$\eta_m=\frac{1-\sqrt{1-4AK/\kappa}}{2A} \tag{26}$$

The expanded form of this solution is

$$\eta_m=\frac{K}{\kappa}+A\left(\frac{K}{\kappa}\right)^2+2A^2\left(\frac{K}{\kappa}\right)^3+\cdots$$

Since the value for K can at least be characterized by the Clauser constant $k\approx 0.016$ at the inner edge of the outer region, it is concluded that η_m is relatively small.

Equilibrium Boundary-Layer Solution

A one-parameter shooting technique can be used to solve this problem. The parameter is f'_m, and the condition to be satisfied is the far-field boundary condition, Eq. (15). A value for f'_m is guessed, and Eq. (26) is solved for η_m. A value for f_m is obtained from Eq. (21), which can be written with the aid of Eq. (23) as

$$f_m = \eta_m\left[f'_m - \frac{1}{\kappa} - \frac{\Pi}{\kappa}\left(\frac{\Delta\eta_m}{\delta}\right)^2\left(4 - 3\frac{\Delta\eta_m}{\delta}\right)\right]$$

The equilibrium form of Eq. (18), which is written as

$$Kf'' - \beta\left(1 + \frac{1}{m}\right)\left[\eta f' - \frac{(1-\gamma f')}{1+\gamma/\kappa}f\right] + 2\beta f - 1$$
$$= -\beta\gamma\left[1 + \left(1 + \frac{1}{m}\right)\frac{1-\gamma/\kappa}{1+\gamma/\kappa}\right]\int_0^\eta (f')^2\,\mathrm{d}\eta$$

is integrated across the boundary layer, and the value of f_∞ is compared with -1. The value of f'_m is iterated until the value of f_∞ converges to -1.

Analytical Solution for $\beta = -1/2$

A closed-form solution exists for the lowest-order equilibrium flow case involving the most favorable pressure gradient ($\beta = -1/2$). The lowest-order governing equation for equilibrium flow with $\beta = -1/2$ and K replaced by the Clauser constant k is

$$kf''_0 - f_0 = 1$$

Equation (24) for Π shows that the wake function vanishes for $\beta = -1/2$. Therefore, Eq. (21) has the form

$$f_{0,m} = \eta_m\left(f'_{0,m} - \frac{1}{\kappa}\right)$$

The equation for η_m is

$$\eta_m = \frac{\sqrt{1 + 4f_{0,m}k/(\eta_m\kappa)} - 1}{2f_{0,m}/\eta_m}$$

The solution is

$$f_0(\eta) = \frac{k}{\kappa\eta_m}\exp\left(\frac{\eta_m - \eta}{k^{1/2}}\right) - 1$$

where

$$\eta_m = 1/2[\kappa - k^{1/2} - \sqrt{(\kappa - k^{1/2})^2 - 4k}]$$

The defect velocity is

$$\frac{u - U}{u^*} = f'_0 = -\frac{k^{1/2}}{\kappa\eta_m}\exp\left(\frac{\eta_m - \eta}{k^{1/2}}\right)$$

and the shear stress velocity is

$$\frac{u^*}{U} = \left[\frac{1}{\kappa}\ell n\left(\frac{U\delta^*}{\nu}\eta_m\right) + B + \frac{k^{1/2}}{\kappa\eta_m}\right]^{-1}$$

Results and Discussion

Present solutions for incompressible, equilibrium boundary layers have been computed with the asymptotic, or zero order, and full-equation forms of the present method. As discussed previously, the asymptotic form is taken in the limit of vanishing shear-stress-velocity to edge-velocity ratio, $\gamma = u^*/U$. All present solutions have been computed with the wake function.

The equilibrium boundary-layer problem is solved easily using a shooting technique. A value of f'_m is iterated until the far-field boundary condition of f is satisfied. A fourth-order Runge-Kutta routine[8] is used to integrate across the outer region on the boundary layer. The empirical parameters are chosen to conform with the method of Mellor and Gibson[2]: the von Karman constants κ and B are given the values 0.41 and 4.9, respectively, and the outer-region eddy viscosity constant k is given the value 0.016.

The present solutions are compared with the full-equation equilibrium solutions of Mellor and Gibson.[2] All of these results are for a Reynolds number based on edge velocity and displacement thickness (Re_{δ^*}) of 1×10^5. The comparison shown in Fig. 1 includes the zero-order closed-form solution for the most favorable pressure gradient, $\beta = -1/2$. The present zero-order numerical solution is seen to be in complete agreement with the closed-form solution and in close agreement with Mellor and Gibson's full-equation solution.

It should be noted in Fig. 1 that the present method has resolved a turbulent boundary layer with only 11 grid points. It is fair to question the accuracy of a solution with so few grid points. Figure 2 compares various grids for a typical β and shows that the 11-point solution is in agreement with the 51- and 101-point solutions. Only the six-point solution deviates noticeably from the fine-grid solutions. On the basis of these observations, the present solutions were computed with 11 uniformly spaced grid points. The first grid point is always at the match (patch) point, and the edge value of η is fixed. During the iteration of f'_m the first grid point, η_m, shifts with f'_m according to Eq. (26). The uniform grid is recomputed for

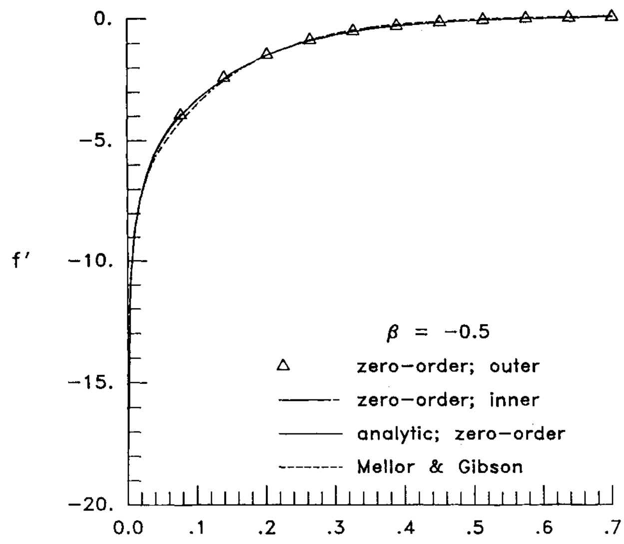

Fig. 1 Defect velocity profiles, $\beta = -0.5$.

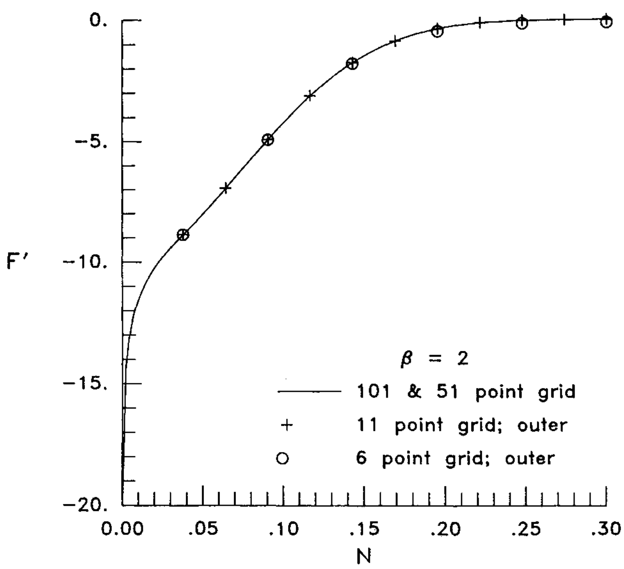

Fig. 2 Grid resolution comparison, $\beta = 2$.

each value of f'_m. This grid procedure was used for convenience rather than necessity as the grid points other than the first could have been fixed. By eliminating the inner-layer computations, a grid-point savings on the order of 50% was realized as was the case in Refs. 1 and 5.

Equilibrium defect profiles are presented in Fig. 3 for a representative small-β case ($\beta \leq 1$); and in Fig. 4, results are presented for a representative large-β case ($\beta > 1$). It can be seen in these figures that the full-equation form of the present method compares extremely well with the full-equation solutions of Mellor and Gibson. Even more important is the excellent agreement of the present zero-order solutions with the full-equation solutions. The simplicity of the zero-order approach and its excellent results make it a desirable alternative to the full-equation approach.

The differences seen in the inner region are caused by the differences between Mellor and Gibson's inner region eddy viscosity model and the present analytic combination of the law of the wall and law of the wake. Mellor and Gibson's solutions approach the law of the wall in the limit as η goes to zero. As will be seen later, Mellor and Gibson's approach allows a large deviation from the law of the wall and wake in the inner region. This effect becomes more noticeable with increasingly strong adverse pressure gradients (increasing β).

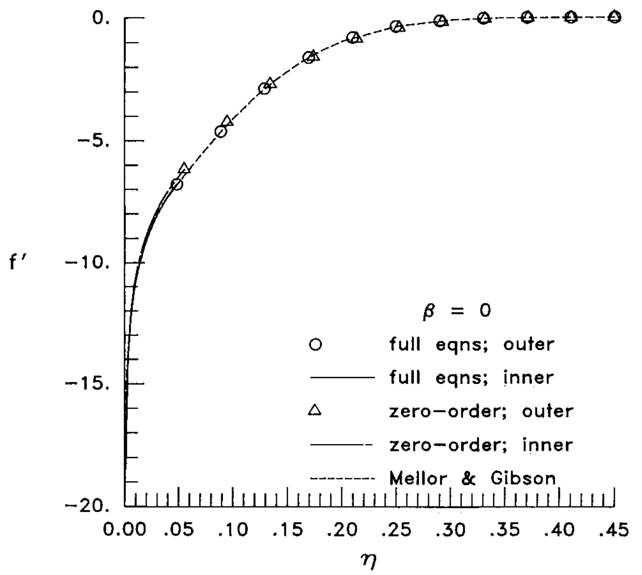

Fig. 3 Defect velocity profiles, $\beta = 0$.

Solutions for values of β greater than about 8.6 cannot be obtained for the values of the Clauser constant and von Karman constants used here unless the size of the law of the wake coefficient Π is reduced. This failure to get a solution is consistent with the observations of Mellor and Gibson and others that Π varies as $\beta^{1/2}$ rather than $\beta^{3/4}$ for large β.

Results for the shear stress velocity to edge velocity ratio, γ, from the present solutions are compared with the results of Mellor and Gibson in Fig. 5. The two sets of results are in close agreement but are not identical.

The present method for evaluating γ differs substantially from that of Refs. 2 and 5 and others in that the value of γ is obtained from a patching of the inner and outer solutions at a particular point in the present method whereas a matching of the outer limit of the inner solution with the inner limit of the outer solution is used in the other methods. These methods are similar to the present method in that inner and outer eddy viscosity models are patched at a point analogous to the present patch point; the matching is done in a region much closer to the wall where the inner eddy viscosity pertains. The equation Mellor and Gibson used to evaluate γ is very similar to Eq. (10), the equation used in the present method. The difference is one of application: Eq. (10) is evaluated at a point a finite distance from the surface in the present method, while the equation used by Mellor and Gibson is evaluated as close to the surface as possible (the inner limit of the outer solution). As a result, the values of γ obtained by Mellor and Gibson are smaller than those obtained by the present method. However, larger values for γ can be obtained from the results of Mellor and Gibson by evaluating their equation at the point where the eddy viscosities are patched.

The inner region velocity profiles of the present method differ from those of Mellor and Gibson. The differences increase with increasingly strong adverse pressure gradient (increasing β) and are best displayed using inner variables (u^+,y^+). Figure 6 shows the $\beta = 4$ case. It is clear in this figure that, although the inner solutions are identical near the wall, Mellor and Gibson's solution deviates from the logarithmic behavior at least an order of magnitude (in y^+ units) closer to the wall than the present solution. In view of this difference, a comparison with experimental data seems appropriate. Figure 7 compares the present solution with experimental data taken from Ref. 9 for Clauser's[3] second equilibrium flow. The data shown in Fig. 7 correspond to profile 2305 in Ref. 9. For this profile, $\beta = 7.531$, and the Reynolds number based on edge velocity and displacement

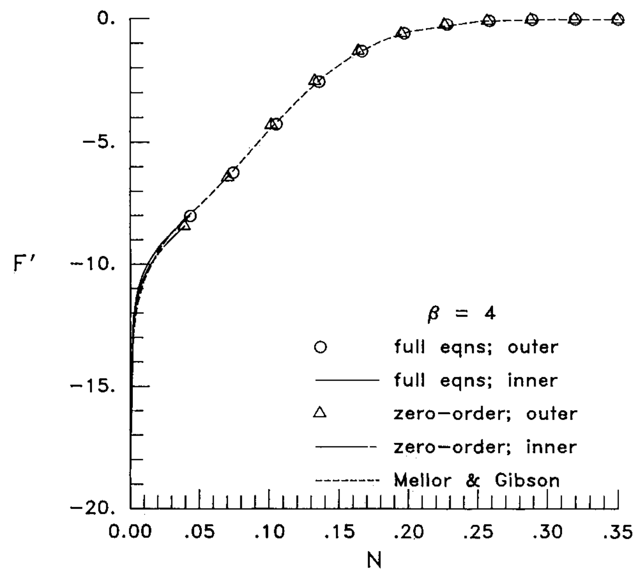

Fig. 4 Defect velocity profiles, $\beta = 4$.

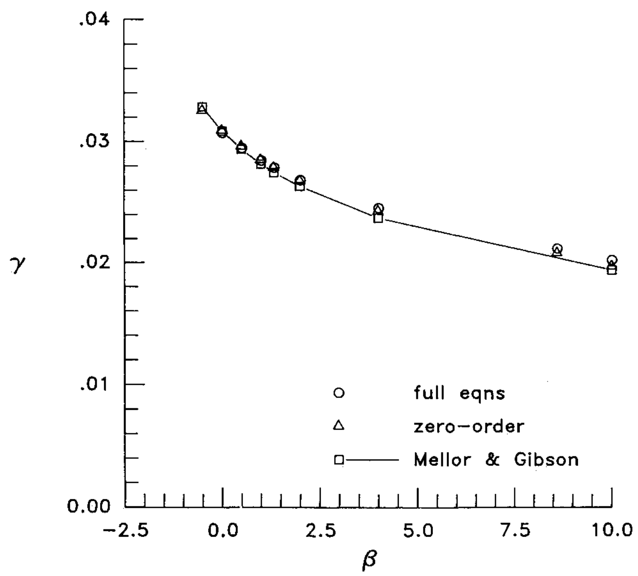

Fig. 5 Skin friction comparison, $Re_{\delta^*} = 10^5$.

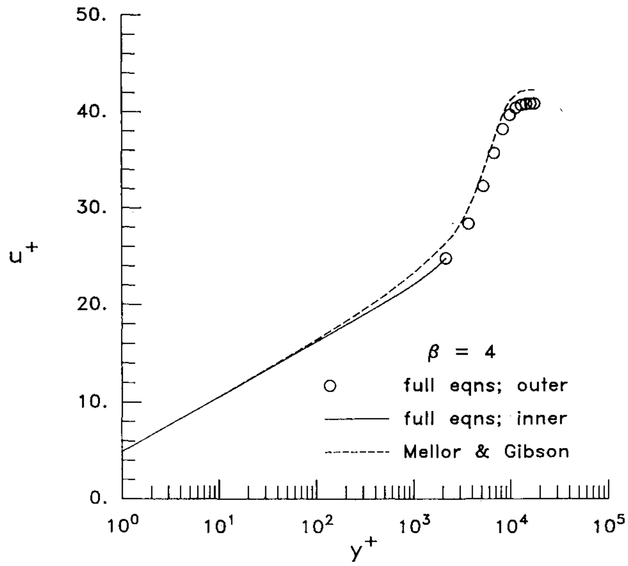

Fig. 6 Inner variable profiles, $\beta = 4$, $Re_{\delta *} = 10^5$.

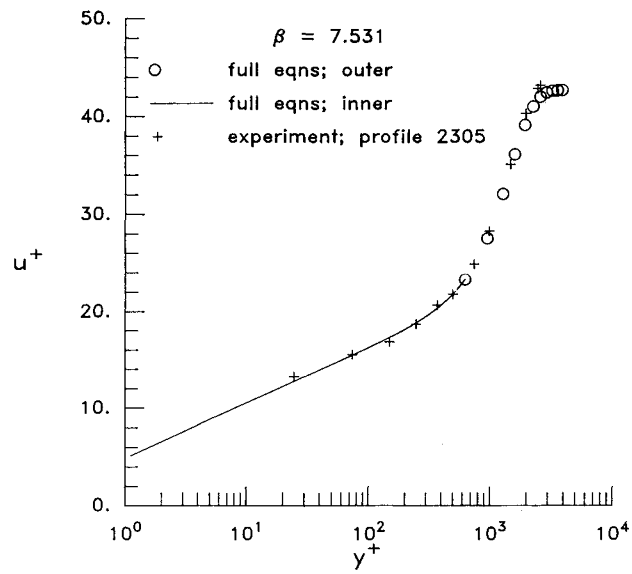

Fig. 7 Inner variable profiles, $\beta = 7.531$, $Re_{\delta *} = 30{,}692.5$.

thickness is 30,692.5. Figure 7 shows excellent agreement between the experimental data and the present solution. Given the large value of β, one would expect less agreement, especially in the logarithmic region, between a Mellor and Gibson solution and experiment. The present solution and experiment were also in close agreement for γ (1.16% difference). Present solutions were computed and compared with experimental data from Ref. 9 for many values of β, and good agreement was generally observed.

Another feature of the present solution can be observed quite well in Fig. 7. Notice that the match point of the present solution is well beyond the logarithmic region. This result demonstrates the necessity for including the wake function in the present formulation. For all present solutions, the wake function allows the match point to move beyond the logarithmic region thus extending the usefulness of the analytic inner layer model and further reducing the region that must be resolved numerically. As expected, the influence of the wake function increased as β increases.

Concluding Remarks

The application of the defect stream function method of Mellor and Gibson[2] to the solution of the two-dimensional, incompressible boundary-layer problem has been reexamined. This formulation is of particular interest to the numerical analyst because it has a zero-order approximation for which the tangential momentum equation has a first integral. The approximate form is obtained in the limit of vanishing shear-stress-velocity to edge-velocity ratio. The present results show that the agreement between the full-equation and zero-order forms of the equilibrium boundary layer equations is excellent for both large and small values of the Clauser pressure gradient parameter.

The lack of popularity of the Mellor and Gibson formulation is probably due to the difficulty encountered in enforcing the no-slip surface boundary condition expressed in terms of the defect stream function. The present method overcomes this difficulty with a law-of-the-wall/law-of-the-wake formulation for the inner part of the boundary layer, which is mathematically patched to the outer formulation and implemented as a boundary condition. This formulation eliminates the need for an eddy viscosity model in the inner region.

The present solutions for equilibrium boundary layers show excellent agreement with previous numerical results and experimental data. In the inner region, where previous numerical results have deviated from experimental data, the present results show good agreement with experiment. Since numerical solutions are not needed in the inner region, the grid required for a given accuracy is reduced by approximately 50%. Converged solutions have been obtained with as few as 11 grid points across the boundary layer. The results show that the use of the wake function permits patch point locations beyond the logarithmic region.

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